Math 564: Real analysis and measure theory
Lecture 1

Motivation for measure theory.

Probability. We understand well the probability theory of n coin tosses, where the probability of 1 is $p \in (0,1)$ and of 0 is $1-p$.

$$\underbrace{\boxed{0/1}\boxed{0/1}\boxed{\;\cdots\;}\boxed{0/1}}_{n}$$

Then for each word $w \in 2^n = \{0,1\}^n$, the probability of coin tosses resulting in w is

$$\mathbb{P}_p(w) = p^{(\#\text{ of } 1\text{s in } w)} \cdot (1-p)^{(\#\text{ of } 0\text{s in } w)}.$$

What if $n = \infty$? In other words, we consider the space $2^{\mathbb{N}} = \{0,1\}^{\mathbb{N}}$ of infinite binary sequences, with the same probabilities of tossing 1 or 0. Then how do we define the probability of "events" in this space?

Geometry. We would like to have a robust notion of volume in $\mathbb{R}^d$, $d \geq 1$, i.e. we would like to determine the volume of a large class of subsets of $\mathbb{R}^d$. We know what the volume of a *box* $B := I_1 \times I_2 \times \dots \times I_d \subseteq \mathbb{R}^d$ should be, where $I_j \subseteq \mathbb{R}$ is an interval:

$$\text{Volume}(B) = \text{lh}(I_1) \cdot \text{lh}(I_2) \cdot \ldots \cdot \text{lh}(I_d),$$

where $\text{lh}(I) :=$ right endpoint $-$ left endpoint. We want to extend this to a class of sets which are closed under ctbl operations: complements, ctbl unions / ctbl intersections.

Analysis. The class of Riemann integral functions in not closed under pointwise limits; indeed, even a pointwise limit of continuous functions on $[0,1]$ is typically not Riemann integrable. But the whole subject of analysis is about approximation/limits, so we would like to extend the class of integrable functions so it becomes closed under ptwise limits. Clearly, for a subset $B \subseteq \mathbb{R}^d$, the integral of its indicator function $\mathbb{1}_B$ will simply be Volume(B), so this task subsumes the previous task about volume.

Polish spaces.

We now define a very robust class of metric spaces that we will be working with throughout and that arises naturally in analysis and related fields.

Def. A metric space (X, d) is called **Polish** if d is a complete metric (every d-Cauchy sequence converges) and X is separable (i.e. there is a ctbl dense set).

Prop. A metric space X is separable $\Leftrightarrow$ it is 2^{nd} ctbl, i.e. admits a ctbl basis of open sets.

Proof: HW.

Recall/learn: In a metric space X, a **basis** is a collection $\mathcal{U}$ of open subsets of X such that every open set is a union (maybe unctbl) of sets in $\mathcal{U}$.

Examples of Polish spaces.

(a) $\mathbb{R}$, or more generally, $\mathbb{R}^d$, with the metric $d_\infty(\vec{x}, \vec{y}) := \max_{i=1}^{d} |x_i - y_i|$.
We know from undergrad. analysis that this is a complete metric. Also, rationals are dense and ctbl, so $\mathbb{Q}^d \subseteq \mathbb{R}^d$ is dense and ctbl.
Note that open intervals with rational endpoints form a ctbl basis for $\mathbb{R}$ and thus open boxes with rational coordinates form a ctbl basis for $\mathbb{R}^d$.

We can also equip $\mathbb{R}^d$ with other equivalent complete metrics (two metrics are equivalent if they induce the same open sets), namely, for $1 \le p < \infty$,

$$d_p(\vec{x},\vec{y}) := \left(\sum_{i=1}^{d} |x_i - y_i|^p\right)^{\frac{1}{p}}.$$

One can show that d_p is bi-Lipschitz equivalent to d_∞, i.e. there is a constant $C_p > 0$ such that

$$\frac{1}{C_p}\cdot d_\infty \le d_p \le C_p \cdot d_\infty.$$

In particular, the spaces $(\mathbb{R}^d, d_p)$ are Polish, for $1 \le p \le \infty$. It's also easy to see that $\lim_{p\to\infty} d_p = d_\infty$. (HW)

(b) If (X,d) is a Polish metric space, then any closed subset is still Polish with the same metric (indeed, closedness ensures completeness of d and any subspace of a 2nd cttl space is 2nd cttl). What about open subsets, say $(0,1)$ in $\mathbb{R}$? The same metric won't work because it won't be complete, but maybe we can take a different equivalent metric that is complete. Indeed, d_∞ on $\mathbb{R}$ is complete and $\mathbb{R}$ looks like $(0,1)$, in other words, they are homeomorphic.

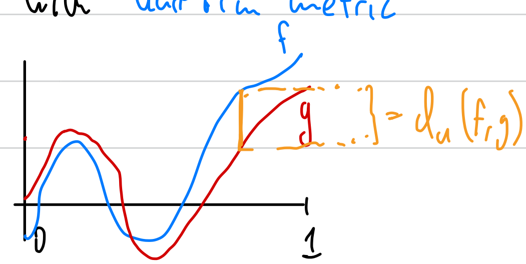
Thus we can "copy" the complete metric from to $(0,1)$ via any homeomorphism. More concretely, $d(x,y) := d_\infty(x,y) + \left|\frac{1}{d_\infty(x,\{0,1\})} - \frac{1}{d_\infty(y,\{0,1\})}\right|$ is a complete metric on $(0,1)$ equivalent to d_∞.

Such sets are called Polishable, and it is a theorem in descriptive set theory that a subset of a Polish space is Polishable if and only if it is G_δ (cttl intersection of open sets).

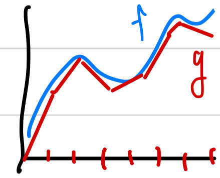
(c) The space $C([0,1])$ of continuous functions on $[0,1]$ with uniform metric

$$d_u(f,g) := \overset{\max}{\sup_{x\in[0,1]}} |f(x) - g(x)|$$

is Polish.

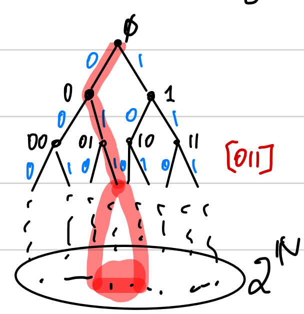


Indeed, we know from undergrad analysis that a uniformly Cauchy sequence of continuous functions converges to a continuous function, so d_u is complete. As for separability, polynomials with rational coefficients form a ctbl dense set (by Weierstrass's theorem), or more easily, piecewise linear functions (with finitely many pieces) with rational breakpoints form a ctbl dense set.



(d) The tree-spaces: Cantor space $2^{\mathbb{N}}$ and Baire space $\mathbb{N}^{\mathbb{N}}$.

Let A be a nonempty ctbl set, e.g. $A := 2 := \{0,1\}$ or $A := \mathbb{N}$. Let $X := A^{\mathbb{N}}$ of infinite sequences of elements of A. We depict $A^{\mathbb{N}}$ as the infinite branches through the tree $A^{<\mathbb{N}} :=$ the set of finite sequences in A:



We equip $A^{\mathbb{N}}$ with the metric: if $x \neq y \in A^{\mathbb{N}}$ then

$$d(x,y) := 2^{-\Delta(x,y)}, \text{ where } \Delta(x,y) := \min i \in \mathbb{N} \text{ with } x_i \neq y_i,$$

and $d(x,y) := 0$ if $x = y$.

This d is indeed a metric on $A^{\mathbb{N}}$, in fact an ultrametric (HW).

Also d is a complete metric (HW) and for a fixed $a_0 \in A$, the set of sequences which are eventually a_0 forms a ctbl dense set. Thus, $A^{\mathbb{N}}$ is Polish.

<u>The topology of $A^{\mathbb{N}}$</u> (the set of open sets). For $2^{-n} < r \leq 2^{-(n-1)}$, the open ball

$$\begin{aligned} B_r(x) &:= \{y \in A^{\mathbb{N}} : d(y,x) < r\} \\ &= \{y \in A^{\mathbb{N}} : d(y,x) \leq 2^{-n}\} = \overline{B}_r(x) \\ &= \{y \in A^{\mathbb{N}} : y|_n = x|_n\}, \text{ where } n = \{0, \dots, n-1\}. \\ &= [x|_n], \end{aligned}$$

where the last term denotes the cylinder with base $x|_n \in A^n$. More generally, for a finite word $w \in A^{<\mathbb{N}}$, let

$$[w] := \{y \in A^{\mathbb{N}} : y \sqsupseteq w\} = \{y \in A^{\mathbb{N}} : y|_{lh(w)} = w\}$$

denote the cylinder with base w. Each cylinder is an open ball, as well as a closed ball, whose center is any element of it (the realtors' metric).

Thus, every open set is a union of cylinders, hence the cylinders form a ctbl basis for $A^{\mathbb{N}}$. When working with $A^{\mathbb{N}}$, we work with this basis. Cylinders are clopen, which makes $A^{\mathbb{N}}$ totally disconnected, in fact, 0-dimensional.

Prop. $A^{\mathbb{N}}$ is compact $\Leftrightarrow$ A is finite.

Proof. Uses König's lemma, HW. $\square$

I prefer $A^{\mathbb{N}}$ to reals $\mathbb{R}^d$ because $A^{\mathbb{N}}$ is so disconnected that it behaves like a discrete space, so we can do combinatorics on it, while still being able to take limits.